Indian Statistical Institute, Bangalore
M. Math.I Year, First Semester
Mid-Sem Examination
Algebra -I

Time: 3 hours September 14, 2009 Instructor: N.S.N.Sastry

Maximum Marks 100

Answer all questions. Your answers should be complete and clearly written. All rings are assumed to contain the multiplicative identity.

1. (a) Show that any integral domain can be considered as a subring of a field.

   (b) Construct a subring of the ring of rational numbers in which $\frac{1}{2}$ is invertible, but $\frac{1}{5}$ is not invertible. [8+4]

2. Determine which of the two polynomials $f(x) \in \mathbb{F}_2[x]$ below has the property that $\mathbb{F}_2[x]/ < f(x) >$ is a field:

   (i) $X^4 + X^3 + 1$ (ii) $X^4 + X^2 + 1$.

   Justify your answer. [8]

3. (a) Define the concept of a t-transitive group.

   (b) For the natural action of the symmetric group S_n on the set X of subsets of $\{1, \cdots, n\}$ of size $n-2$ each, find the largest t for which the above action is t-transitive. [4+4]

4. (a) Define the concepts of degree' and primitivity' for a transitive permutation group.

   (b) If G acts transitively on a set X and H is the stabilizer of a point of X, then show that the action is primitive if, and only if, H is a maximal subgroup of G. [3+3+8]

5. Let R be the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and let M be the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$ such that the following folds:

   Given any $\varepsilon > 0$, there exists a real number $N_\varepsilon \geq 0$ such that $|x| \geq N_\varepsilon$ implies that $|f(x)| \leq \varepsilon$.

   Is M a maximal ideal of R ? If it is not, find a maximal ideal of R containing M. [8+8]

6. (a) Let G be a group of order $P^n m$, where p is a prime coprime to m. Let H b a subgroup of G of order $p^r, r < n$. Show that H is contained in a subgroups of G of order p^n.

   (b) Show that any group of order 35 is cyclic.

(c) Does there exist a noncyclic group of order 21?. If not, give an example. [8+8+8]

7. Let G be group of order p^n, p prime, acting on a finite dimensional vector space V over a field of p elements. Show that there exists a non zero element of V which is fixed by each element of G. [8]

8. (a) Define the group ring $k[G]$ of a group G over a field k.

   (b) If C is a nontrivial conjugacy class of elements of G, show that the sum of the elements of C is contained in the centre of the group ring $k[G]$.

[8+4]

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